Energy dissipation for quasielastic granular particle collisions

W. A. M. Morgado
Department of Physics, MIT, Cambridge, Massachusetts 02139

I. Oppenheim
Department of Chemistry, MIT, Cambridge, Massachusetts 02139

(Received 6 August 1996)

From a first principles theory for the behavior of smooth granular systems, we derive the form for the instantaneous dissipative force acting between two grains. The present model, which is based on the classical harmonic crystal, reproduces the dependence of the kinetic energy dissipation on the grain deformation obtained by models that assume a viscoelastic behavior (without permanent plastic deformations) during the collision. [S1063-651X(97)04902-7]

PACS number(s): 46.10.+z, 05.60.+w, 46.30.Pa, 51.10.+y

I. INTRODUCTION

The term granular system designates a wide range range of physical systems that are characterized by certain common features [1]: they exist in macroscopic portions (grains), which, in the dry state, interact mainly repulsively through rigid elastic interactions; there is contact friction between the grains; and energy is dissipated during collisions due to the excitation of the internal modes. Sand, powders, particles in planetary rings [2], salt and sugar, and grains in a silo [3] are just a few of the many examples of granular systems. These systems are extremely important for a number of industrial applications [4] (transport properties of powder or grains, mixtures of grains and fluids, etc.). Thus it is important to understand their flow properties [1,5,6]. The loss of kinetic energy via heating the grains makes the granular gas fundamentally different from a molecular system [7] and an important medium to study nonequilibrium phenomena. Granular systems present a rich variety of behavior ranging from solidlike to liquidlike, depending on the external stresses applied to the system [8].

The inelastic character of granular collisions can be summarized in terms of a coefficient of restitution which is a proportionality relation between the final and initial relative velocities in a collision. Even though the coefficient is known to depend on the initial relative velocity [9], for simplicity and computational economy, several authors have assumed it to be independent of the collision parameters [1,5,6]. In order to improve the understanding of instantaneous energy dissipation, some authors have tried to model relative velocity-dependent dissipation functions. Self-consistent dimensional arguments were used [4], but a more promising line assumes that the collision is slow enough so that a weak viscoelastic frictional force is superimposed onto the elastic repulsive force [10,11].

In the present study, we propose to investigate a simple microscopic model for harmonic grains. Under the assumption that the collisions are slow enough so that no plastic deformations occur, the present model reproduces the results obtained using a viscoelastic interaction between the grains [10,11].

The paper is organized as follows. In Sec. II, we develop the model for the potential energy of deformed harmonic grains from the classical harmonic crystal model. In Sec. III, we study the case of two interacting grains. In Sec. IV, a transport coefficient describing the instantaneous kinetic energy dissipation is obtained. In Sec. V, we obtain the equation of motion for colliding grains. In Sec. VI, the magnitude of the dissipative term is evaluated. In Sec. VII, we present a brief summary of our results.

II. DEFORMATIONS AND POTENTIAL ENERGY

In this section we derive the potential energy of a deformed spherical granular particle. We assume that its atoms interact through microscopic two-body potentials. The atoms in the granular particles (GP) are arranged in a crystalline lattice form. Let $\mathbf{R}_i$ denote the equilibrium, nondeformed position of atom i ($\mathbf{R}_i \equiv a\hat{\mathbf{a}} + b\hat{\mathbf{b}} + c\hat{\mathbf{c}}$, where $\hat{\mathbf{a}},\hat{\mathbf{b}},\hat{\mathbf{c}}$ are the Bravais lattice basis vectors, and a,b,c are integers) [12]. Let $\boldsymbol{\rho}_i$ denote the displacement from equilibrium for atom i, and let $\mathbf{u}_i \equiv \mathbf{u}(\mathbf{R}_i)$ denote the displacement of atom i's equilibrium position due to external constraints. The atom's actual position $\mathbf{d}_i$ is therefore given by

$$\mathbf{d}_i = \mathbf{R}_i + \mathbf{u}_i + \boldsymbol{\rho}_i .$$

The distance between two atoms in the deformed medium is then a function of the initial undeformed distance and of the elastic and thermal displacements,

$$\mathbf{d}_{ij} = (\mathbf{R}_j - \mathbf{R}_i) + (\mathbf{u}_j - \mathbf{u}_i) + (\boldsymbol{\rho}_j - \boldsymbol{\rho}_i). \tag{1}$$

The difference $\mathbf{u}_j - \mathbf{u}_i$ depends on the deformation experienced by the crystal. If $\mathbf{R}_j - \mathbf{R}_i$ is small compared to the characteristic length associated with the deformations, we obtain

$$\mathbf{u}_j - \mathbf{u}_i \approx (\mathbf{R}_j - \mathbf{R}_i)\cdot \nabla_{\mathbf{R}} \mathbf{u}(\mathbf{R})\big|_{\mathbf{R}=\mathbf{R}_i}$$
$$= (\mathbf{R}_j - \mathbf{R}_i)_\beta \frac{\partial}{\partial x_\beta} \mathbf{u}(\mathbf{x})\bigg|_{\mathbf{x}=\mathbf{R}_i}, \tag{2}$$

where Greek subscripts denote coordinate indices and repeated indices imply summation (for $\beta=1,2,3$). The distance between atoms i and j is then given by

$$\mathbf{d}_{ij}=(\mathbf{R}_j-\mathbf{R}_i)+(\mathbf{R}_j-\mathbf{R}_i)_\beta\frac{\partial}{\partial x_{i\beta}}\mathbf{u}(\mathbf{R}_i)+(\boldsymbol{\rho}_j-\boldsymbol{\rho}_i). \quad (3)$$

We will assume that effects associated with displacement of dislocations and other crystal defects are small. Thus, we can expand the total crystal potential energy around the new positions of the atoms up to second order in the variations of the positions. By expanding the potential

$$\Phi=\frac{1}{2}\sum_i\sum_{i\neq j}\phi_{ij}$$

around the equilibrium distance of $(\mathbf{R}_j-\mathbf{R}_i)$ [13], we obtain

$$\phi_{ij}=\phi(|\mathbf{d}_{ij}|),$$

$$\phi_{ij}\approx\phi(|\mathbf{R}_j-\mathbf{R}_i|)+(\boldsymbol{\beta}_{ij}\cdot\nabla)\phi(|\mathbf{R}_j-\mathbf{R}_i|)+\tfrac{1}{2}(\boldsymbol{\beta}_{ij}.\nabla)^2\phi(|\mathbf{R}_j-\mathbf{R}_i|),$$

$$\Phi\approx\frac{1}{2}\sum_{i,j}\phi(|\mathbf{R}_j-\mathbf{R}_i|)+\frac{1}{4}\sum_{i,j}(\boldsymbol{\beta}_{ij}.\nabla)^2\phi(|\mathbf{R}_j-\mathbf{R}_i|), \quad (4)$$

up to quadratic order, where $\boldsymbol{\beta}_{ij}=\boldsymbol{\beta}_j-\boldsymbol{\beta}_i=(\mathbf{R}_j-\mathbf{R}_i)\cdot\nabla\mathbf{u}+(\boldsymbol{\rho}_j-\boldsymbol{\rho}_i)$, and the indices i and j run over all atoms in the GP.

We define the matrix $\mathbf{D}^{ij}_{\mu\nu}$

$$\mathbf{D}^{ij}_{\mu\nu}=\delta_{i,j}\sum_k\frac{\partial^2\phi_{ik}}{\partial x_\mu\partial x_\nu}-\frac{\partial^2\phi_{ij}}{\partial x_\mu\partial x_\nu}. \quad (5)$$

With the properties

$$\mathbf{D}^{ij}_{\mu\nu}=\mathbf{D}^{ji}_{\nu\mu},\quad \mathbf{D}^{ij}_{\mu\nu}=\mathbf{D}^{ij}_{\nu\mu},\quad \sum_i^{\text{all atoms}}\mathbf{D}^{ij}_{\mu\nu}=0.$$

Thus Φ can be written in the form

$$\Phi=\frac{1}{2}\sum_{i,j}\phi(|\mathbf{R}_j-\mathbf{R}_i|)+\frac{1}{2}\sum_{i,j}\beta_{i\mu}\mathbf{D}^{ij}_{\mu\nu}\beta_{j\nu}. \quad (6)$$

The potential Φ can be separated into ground state energy, harmonic potential, elastic potential, and coupling parts as follows:

$$\Phi=\Phi_0+\frac{1}{2}\sum_{i,j}\rho_{i\mu}\mathbf{D}^{ij}_{\mu\nu}\rho_{j\nu}+\frac{1}{2}\sum_{i,j}R_{i\alpha}\mathbf{D}^{ij}_{\mu\nu}R_{j\beta}\frac{\partial u_\mu}{\partial x_\alpha}\frac{\partial u_\nu}{\partial x_\beta}+\sum_{i,j}R_{i\alpha}\mathbf{D}^{ij}_{\mu\nu}\frac{\partial u_\mu}{\partial x_\alpha}\rho_{j\nu}, \quad (7)$$

where $\Phi_0=\Sigma_j\Sigma_{i<j}\phi(|\mathbf{R}_j-\mathbf{R}_i|)$.

The elastic potential term (due to the deformation) which is given by

$$V_{el}=\frac{1}{2}\sum_{i,j}R_{i\alpha}\mathbf{D}^{ij}_{\mu\nu}R_{j\beta}\frac{\partial u_\mu}{\partial x_\alpha}(\mathbf{R}_i)\frac{\partial u_\nu}{\partial x_\beta}(\mathbf{R}_j),$$

accounts for the classical elastic energy of the material [12]. Using the properties of $\mathbf{D}^{ij}_{\mu\nu}$ we can rewrite V_{el} in the form

$$V_{el}=\frac{1}{2}\sum_j E_{\alpha\mu\beta\nu}\frac{\partial u_\mu}{\partial x_\alpha}(\mathbf{R}_j)\frac{\partial u_\nu}{\partial x_\beta}(\mathbf{R}_j),$$

where the derivatives of $\mathbf{u}$ are taken at the point $\mathbf{R}_i$ and

$$E_{\alpha\mu\beta\nu}=-\frac{1}{2}\sum_i^{\text{all atoms}}R_{i\alpha}\mathbf{D}_{\mu\nu}(\mathbf{R}_i)R_{i\beta}.$$

The above expression can be further transformed by noticing that a pure rotation does not change the solid energy. This implies that V_{el} can depend on $(\partial u_\nu/\partial x_\beta)$ only through the strain tensor, which for small deformations has the form

$$u_{\alpha\beta}=\frac{1}{2}\left(\frac{\partial u_\alpha}{\partial x_\beta}+\frac{\partial u_\beta}{\partial x_\alpha}\right). \quad (8)$$

The elastic term then becomes [12]

$$V_{el}=\frac{1}{2}\sum_j E_{(\alpha\mu)(\beta\nu)}u_{\alpha\mu}(\mathbf{R}_j)u_{\nu\beta}(\mathbf{R}_j), \quad (9)$$

where the symbols $(\alpha\mu)$ and $(\beta\nu)$ denote the symmetric part of $E_{(\alpha\mu)(\beta\nu)}$ with respect to $(\alpha\mu)$ and $(\beta\nu)$, respectively.

The term

$$V_\phi=\sum_{i,j}R_{i\alpha}\mathbf{D}^{ij}_{\mu\nu}\frac{\partial u_\mu}{\partial x_\alpha}(\mathbf{R}_i)\rho_{j\nu},$$

is responsible for the coupling between the internal degrees of freedom, $\boldsymbol{\rho}_i$, and the distance between the two GPs centers of mass, through the derivative $(\partial u_\mu/\partial x_\alpha)$, which can be taken at $\mathbf{R}_j$ to a very good approximation.

Due to a similar argument to Eq. (9), we can write the coupling term as

$$V_\phi=\sum_j E_{(\alpha\mu)\nu}u_{\alpha\mu}\rho_{j\nu}, \quad (10)$$

where $E_{(\alpha\mu)\nu}$ is the symmetric tensor (on $\alpha\mu$) given by

$$E_{(\alpha\mu)\nu}=-\frac{1}{4}\sum_j[R_{j\alpha}\mathbf{D}_{\mu\nu}(\mathbf{R}_j)+R_{j\mu}\mathbf{D}_{\alpha\nu}(\mathbf{R}_j)]. \quad (11)$$

III. INTERACTION BETWEEN TWO GRANULAR PARTICLES

When two GPs interact, the results of Sec. II must be generalized to include cross-terms between the degrees of freedom of the two grains. We have to take into account the terms that correspond to direct interaction between GPs 1 and 2. Before doing the expansion for ϕ_{ij} (where i belongs to particle 1 and j belongs to particle 2), we can eliminate those ϕ_{ij}'s which are small (those for which $|\mathbf{r}_{ij}|\geqslant r_0\equiv$ interaction range of the order of the lattice parameter). Since this is a short range interaction, only those atoms close to the interface between particles 1 and 2 will contribute. Their number being proportional to the area of the interface, we

will neglect them in comparison with the bulk contribution to the coupling energy. The coupling potential can then be written as (where the superscripts denote the grain)

$$V_\phi=\sum_j^{(1)} E^{(1)}_{(\alpha\mu)\nu}u^{(1)}_{\alpha\mu}\rho^{(1)}_{j\nu}+\sum_j^{(2)} E^{(2)}_{(\alpha\mu)\nu}u^{(2)}_{\alpha\mu}\rho^{(2)}_{j\nu}, \quad (12)$$

where the superscripts (1) and (2) denote particles 1 and 2, and the strain tensor depends on the Bravais summation index. We are going to use the convention that whenever there is a sum over Bravais vectors in the following, the strain tensor depends on the summation index.

The total elastic contribution will be

$$V_{el}=\frac{1}{2}\sum_i^{(1)} E^{(1)}_{(\alpha\mu)(\beta\nu)}u^{(1)}_{\alpha\mu}u^{(1)}_{\nu\beta}+\frac{1}{2}\sum_i^{(2)} E^{(2)}_{(\alpha\mu)(\beta\nu)}u^{(2)}_{\alpha\mu}u^{(2)}_{\nu\beta}, \quad (13)$$

and the total harmonic contribution is given by

$$V_H=\frac{1}{2}\sum_{i,j}^{(1)} (\rho^{(1)}_{j\mu}-\rho^{(1)}_{i\mu})\mathbf{D}^{(1)ij}_{\mu\nu}(\rho^{(1)}_{j\nu}-\rho^{(1)}_{i\nu}) + \frac{1}{2}\sum_{i,j}^{(2)} (\rho^{(2)}_{j\mu}-\rho^{(2)}_{i\mu})\mathbf{D}^{(2)ij}_{\mu\nu}(\rho^{(2)}_{j\nu}-\rho^{(2)}_{i\nu}). \quad (14)$$

IV. DISSIPATION OF ENERGY

Schofield and Oppenheim [14] proposed a mechanism for the dissipation of energy that occurs when two GP (1 and 2) collide. It is mediated by a transport coefficient defined as follows:

$$\gamma(r_{12})=\int_0^\infty d\tau\langle\hat{\nabla V}_\phi e^{-L_{in}\tau}\hat{\nabla V}_\phi\rangle_f, \quad (15)$$

where $\langle\cdots\rangle_f$ means a statistical average keeping the center of masses fixed, L_{in} is the Liouvillian operator for the internal granular degrees of freedom only, (i.e., center of mass positions and total momentum are kept fixed), and $\hat{A}=A-\langle A\rangle_f$. The typical variation of $|\boldsymbol{\rho}_i|$ is smaller than a, the typical lattice spacing. Given that the typical macroscopic deformation is h, and σ is the GP's diameter, we have the inequalities $a\ll h\ll\sigma$.

The coefficient $\gamma(r_{12})$ will be given to lowest order in the strain tensor (which is assumed to be small) by [14]

$$\gamma(r_{12})=\int_0^\infty d\tau\langle\hat{\nabla V}_\phi e^{-L_{harm}\tau}\hat{\nabla V}_\phi\rangle_f.$$

The linearity of V_ϕ in $\boldsymbol{\rho}_i$ leads to $\hat{\nabla V}_\phi=\nabla V_\phi$, due to the symmetry of a harmonic oscillator. Cross averages like $\langle\boldsymbol{\rho}^{(1)}_j\boldsymbol{\rho}^{(2)}_k\rangle_f$ will also vanish. Thus, we have for $\gamma(r_{12})$

$$\gamma(r_{12})=\sum_{i,j}^{(1)} E^{(1)}_{(\alpha\mu)\nu}E^{(1)}_{(\beta\gamma)\theta}(\nabla_{12}u^{(1)}_{\beta\gamma})(\nabla_{12}u^{(1)}_{\alpha\mu}) \times\int_0^\infty d\tau\langle\rho^{(1)}_{i\nu}\rho^{(1)}_{j\theta}(\tau)\rangle_f + \sum_{i,j}^{(2)} E^{(2)}_{(\alpha\mu)\nu}E^{(2)}_{(\beta\gamma)\theta}(\nabla_{12}u^{(2)}_{\beta\gamma})(\nabla_{12}u^{(2)}_{\alpha\mu}) \times\int_0^\infty d\tau\langle\rho^{(2)}_{i\nu}\rho^{(2)}_{j\theta}(\tau)\rangle_f, \quad (16)$$

where $\nabla_{12}=\nabla_{\mathbf{r}_{12}}$.

The harmonic form for the potential gives in the first approximation [15,16]

$$\langle\rho_{i\nu}\rho_{j\theta}(\tau)\rangle_f=\mathbf{D}^{-1ij}_{\nu\theta}k_BTF(\tau), \quad (17)$$

where the correlation function $F(\tau)$ can be assumed to decay quickly for a large range of interatomic potentials to insure the convergence of the time integrals. We will assume the form [16]

$$F(\tau)=e^{-\tau/\tau_v},$$

where τ_v sets the vibrational time scale. The time integrals in Eq. (16) become $\tau_v\mathbf{D}^{-1ij}_{\nu\theta}kT$.

Since the particles are identical, we can rewrite $\gamma(r_{12})$ as

$$\gamma(r_{12})=\tau_v\sum_{i,j} E_{(\alpha\mu)\nu}E_{(\beta\gamma)\theta}\mathbf{D}^{-1ij}_{\nu\theta}\times((\nabla_{12}u^{(1)}_{\beta\gamma})(\nabla_{12}u^{(1)}_{\alpha\mu}) + (\nabla_{12}u^{(2)}_{\beta\gamma})(\nabla_{12}u^{(2)}_{\alpha\mu})). \quad (18)$$

Equation (18) describes the energy dissipation occurring during the collision of two GPs, as will be seen in Sec. V.

V. DISSIPATIVE EQUATION OF MOTION

When two identical GPs of mass m collide, kinetic energy is dissipated through excitation of the internal modes of vibration. In our simple model we neglect plastic deformation effects. Since the collision time is much longer than the vibrational time ($\tau_c\gg\tau_v$), we can assume that the interaction between two GPs is very well described by the elastic Hertzian quasi-static force [17]. The form for the interparticle interaction can also be obtained from a Fokker-Planck equation for the probability distribution $W(X_t,t)$ of a granular system [14]

$$\dot{W}(X_t,t)=\left[\left(-\sum_{i=1}^N\frac{\mathbf{p}_i}{m}\cdot\nabla_{r_i}+\sum_{i=1}^N\nabla_{r_i}(U+\omega)\cdot\nabla_{\mathbf{p}_i}\right) + \frac{1}{2}\sum_j^N\sum_{k\neq j}^N\gamma_{jk}\hat{\mathbf{r}}_{jk}\hat{\mathbf{r}}_{jk}:(\nabla_{\mathbf{p}_j}-\nabla_{\mathbf{p}_k}) \times\left((\nabla_{\mathbf{p}_j}-\nabla_{\mathbf{p}_k})+\beta\frac{\mathbf{p}_j-\mathbf{p}_k}{m}\right)\right]W(t), \quad (19)$$

where $\mathbf{p}_i,\mathbf{r}_i,i=1,\ldots,N$ are the momenta and positions, respectively, of the centers of mass of the grains,

$X_t=\{\mathbf{p}^N,\mathbf{r}^N\}$, $\beta=1/k_BT$, where T is the internal temperature of the GPs, $\gamma_{jk}=\gamma(r_{jk})$ is the dissipative coefficient calculated in Sec. IV and $\hat{\mathbf{r}}_{jk}$ is the unit vector in the direction between particles j and k, and $U+\omega$ is the effective interaction potential between the grains.

In the case of two particles, we define $\mathbf{r}_{1,2}(t)=\langle\mathbf{r}_{1,2}\rangle_t$ and $\mathbf{v}_{1,2}(t)=\langle\dot{\mathbf{r}}_{1,2}\rangle_t$, where $\langle B\rangle_t=\int dX_t W(t)B$. Assuming that $W(X_t,t)$ is sharply peaked, we can approximate $\mathbf{v}_{1,2}(t)\approx\dot{\mathbf{r}}_{1,2}(t)$. We also define relative and center of mass coordinates

$$\mathbf{r}_{12}=\mathbf{r}_2-\mathbf{r}_1,\quad \mathbf{r}_{cm12}=\tfrac{1}{2}(\mathbf{r}_2+\mathbf{r}_1).$$

The deformation parameter h is defined as

$$h=\begin{cases}\sigma-r_{12} & \text{if } \sigma\geqslant r_{12}\\ 0 & \text{otherwise,}\end{cases}$$

where σ is the diameter of a GP.

From the basic result

$$\frac{d}{dt}\langle B\rangle_t=\int dX_T\dot{W}B,$$

and Eq. (19), after assuming that $U+\omega\approx V_{12}^{el}$ we obtain

$$\begin{aligned}\langle\dot{\mathbf{p}}_2\rangle_t-\langle\dot{\mathbf{p}}_1\rangle_t=&\int dX_t(\mathbf{p}_2-\mathbf{p}_1)(\nabla_{\mathbf{r}_1}V_{12}^{el}\cdot\nabla_{\mathbf{p}_1}\\&+\nabla_{\mathbf{r}_2}V_{12}^{el}\nabla_{\mathbf{p}_2})W(t)+\int dX_t\gamma_{12}\hat{\mathbf{r}}_{12}\\&\times(\mathbf{p}_2-\mathbf{p}_1)\hat{\mathbf{r}}_{12}:(\nabla_{\mathbf{p}_1}-\nabla_{\mathbf{p}_2})\Big((\nabla_{\mathbf{p}_1}-\nabla_{\mathbf{p}_2})\\&+\beta\frac{\mathbf{p}_1-\mathbf{p}_2}{m}\Big)W(t)\\=&-\int dX_t W(t)(\nabla_{\mathbf{r}_2}V_{12}^{el}-\nabla_{\mathbf{r}_1}V_{12}^{el})\\&-\beta\int dX_t\gamma_{12}\hat{\mathbf{r}}_{12}\hat{\mathbf{r}}_{12}.\frac{\mathbf{p}_1-\mathbf{p}_2}{m}W(t)\\=&\langle\mathbf{F}_{12}^{el}\rangle_t-\frac{\beta}{m}\langle\gamma_{12}\hat{\mathbf{r}}_{12}\hat{\mathbf{r}}_{12}\cdot(\mathbf{p}_1-\mathbf{p}_2)\rangle_t,\end{aligned}$$

where $\mathbf{F}_{12}^{el}=-V_{12}^{el}(h)'\hat{\mathbf{r}}_{12}$.

The elastic potential between the grains is a function of the tensor $E_{(\alpha\mu)(\beta\gamma)}$ defined in Eq. (9). The microscopic summation form of V_{12}^{el} reduces to the usual macroscopic elastic energy derived from phenomenological theories. In terms of the particles Young modulus E, Poisson ratio Σ, and radius $\sigma/2$, $V_{12}^{el}(h)$ in Eq. (13) is given by [10,l8]

$$V_{12}^{el}(h)=\tfrac{1}{2}kh^{5/2},\tag{20}$$

where for three-dimensional isotropic spheres [17]

$$k=\left(\frac{8}{15\sqrt{2}}\right)\frac{E}{1-\Sigma^2}R^{1/2}.$$

The equation for the time variation of h for a frontal collision becomes

$$\frac{m}{2}\frac{d^2h}{dt^2}=-\frac{5}{4}kh^{\frac{3}{2}}-\frac{\beta\gamma(h)}{m}\frac{dh}{dt}.\tag{21}$$

In the equation above, we observe that the energy dissipation is described by a frictional force mediated by $\gamma(r_{12})$

$$\mathbf{F}_f=-\frac{\beta\gamma(r_{12})}{m}(\hat{\mathbf{k}}\cdot\mathbf{p}_{12})\hat{\mathbf{k}}.$$

We will next evaluate $\gamma(r_{12})\equiv\gamma(h)$ as a function of h so that way we can solve Eq. (21) and thus determine the velocity dependent coefficient of restitution for a quasielastic collision.

VI. THE FRICTIONAL COEFFICIENT γ

From Eq. (18), the expression for $\gamma(h)$ when the particles are identical, is given by

$$\begin{aligned}\gamma(h)=\sum_j F_{(\alpha\mu)(\beta\gamma)}[&(\nabla_{12}u_{\beta\gamma}^{(1)})(\nabla_{12}u_{\alpha\mu}^{(1)})+(\nabla_{12}u_{\beta\gamma}^{(2)})\\&\times(\nabla_{12}u_{\alpha\mu}^{(2)})],\end{aligned}\tag{22}$$

where $F_{(\alpha\mu)(\beta\gamma)}=\tau_v kTE_{(\alpha\mu)\nu}E_{(\beta\gamma)\theta}\mathbf{D}_{\nu\theta}^{-1ij}$

The behavior of $\gamma(h)$ can be obtained by a scaling argument following Landau and Lifshitz [17]. The equation for the radius of the contact region, b, as a function of the curvature of the particle, $A=2/\sigma$, where σ is the diameter of the particles, is given by [17]

$$A=\frac{FD}{\pi}\int_0^\infty\frac{d\xi}{(b^2+\xi)^2(\xi)^{1/2}}.$$

The deformation h is given by [17]

$$h=\frac{FD}{\pi}\int_0^\infty\frac{d\xi}{(b^2+\xi)(\xi)^{1/2}},$$

where F is the force between the spheres and [17]

$$D=\frac{3}{2}\left(\frac{1-\Sigma^2}{E}\right).$$

In order to keep the above equations invariant, we scale lengths by a factor $\sqrt{\alpha}$ ($b\to\sqrt{\alpha}b$, $\xi\to\alpha\xi$ and $\mathbf{x}\to\sqrt{\alpha}\mathbf{x}$), and the force by a factor $\alpha^{3/2}$ ($F\to\alpha^{3/2}F$). We then obtain for $h\to\alpha h$, showing that the force varies as $h^{3/2}$ [ll]. The z component (collision axis) for the deformation vector $u_z(\mathbf{x})$ is given by [19]

$$\begin{aligned}u_z(\mathbf{x})=&\frac{3F}{2\pi a^2}\int_{x'^2+y'^2\leqslant b^2}\frac{dx'dy'}{\sqrt{(x-x')^2+(y-y')^2+z^2}}\\&\times\left(1-\frac{x'^2+y'^2}{b^2}\right)^{1/2}.\end{aligned}\tag{23}$$

Using the variable $\mathbf{x}''=b\mathbf{x}'$, we show that $u_z(\mathbf{x})$ scales as

$$u_z(\mathbf{x}) \rightarrow u_z(\sqrt{\alpha}\mathbf{x}) = \alpha u_z(\mathbf{x}). \tag{24}$$

From above, we obtain the scaling behavior of the strain tensor

$$u_{\alpha\beta}(\mathbf{x}) \rightarrow u_{\alpha\beta}(\sqrt{\alpha}\mathbf{x}) = \sqrt{\alpha} u_{\alpha\beta}(\mathbf{x}). \tag{25}$$

The tensors $E_{(\alpha\mu)(\beta\gamma)}$ and $F_{(\alpha\mu)(\beta\gamma)}$ are invariant under the α scale transformation. The derivatives of the strain tensor scale as

$$\frac{\partial u_{\alpha\beta}(\mathbf{x})}{\partial h} \rightarrow \frac{\partial u_{\alpha\beta}(\sqrt{\alpha}\mathbf{x})}{\alpha \partial h} = \frac{1}{\sqrt{\alpha}} \frac{\partial u_{\alpha\beta}(\mathbf{x})}{\partial h}. \tag{26}$$

We deduce from this that the total elastic energy scales as $\alpha^{5/2}$ and the dissipative function $\gamma(h)$ as $\alpha^{1/2}$, implying that $V_{el} \propto h^{5/2}$ and $\gamma(h) \propto h^{1/2}$.

Due to the scaling properties of the strain tensor $u_{\alpha\beta}$ and its derivative $[\partial u_{\alpha\beta}(\mathbf{x})/\partial h]$, we can write

$$\frac{\partial u_{\alpha\beta}(\mathbf{x})}{\partial h} = \frac{B}{h} u_{\alpha\beta}(\mathbf{x}),$$

where B is a proportionality constant.

We deduce from the above and Eq. (22) that

$$\gamma(h) = \sum_j \mathcal{F}_{(\alpha\mu)(\beta\gamma)} (u^1_{\beta\gamma} u^1_{\alpha\mu} + u^2_{\beta\gamma} u^2_{\alpha\mu}) h^{-2}, \tag{27}$$

where $\mathcal{F}_{(\alpha\mu)(\beta\gamma)} = B^2 F_{(\alpha\mu)(\beta\gamma)}$. The tensor $\mathcal{F}_{(\alpha\mu)(\beta\gamma)}$ possesses the same symmetry properties as $E_{(\alpha\mu)(\beta\gamma)}$. Consequently, $\gamma(h)$ will be a function of different phenomenological parameters but of the same form as $V_{el} \equiv V_{el}(E, \Sigma)$.

Given that the elastic force between the GP is obtained by taking the derivative of V_{el} Eq. (20) with respect to h, we obtain for $\gamma(h)$ in Eq. (22)

$$\gamma(h) = \tfrac{5}{2} k' h^{5/2} h^{-2} = \tfrac{5}{2} k' h^{1/2}, \tag{28}$$

where k' is a function of the tensor $\mathcal{F}_{(\alpha\mu)(\beta\gamma)}$, and has the same form as k with $E \rightarrow \eta_1$ and $\Sigma \rightarrow \eta_2$, where η_1 and η_2 are the corresponding viscoelastic coefficients.

The equation of motion for the GP on a frontal collision then becomes

$$\frac{m}{2} \frac{d^2 h}{dt^2} = \frac{5}{4} k h^{3/2} - \frac{5}{2} k' h^{1/2} \frac{dh}{dt}. \tag{29}$$

Equation (29) reproduces results obtained by heuristic methods [10,11].

The above equation will describe the behavior of a smooth spherical GP at speeds much smaller than the speed of sound inside the grain.

The relative importance of dissipation can be obtained by assuming typical values for the physical parameters in question. The typical interatomic potential is denoted by ϕ_0, the typical value of $u_{\alpha\beta}$ is u, the typical elastic energy is $\phi \approx \Sigma \phi_0 u u$, the typical lattice spacing is a. We may assume $\phi \approx k T_g$ (hard-sphere approximation), where T_g is the granular temperature, and $m v_g^2 \approx k T_g$.

The coupling tensors are given by

$$\mathbf{D}_{\mu\nu} \approx \frac{\phi_0}{a^2}, \quad \mathbf{E}_{(\mu\nu)(\alpha\beta)} \approx \phi_0,$$

thus showing why $\phi \approx \Sigma \phi_0 u u$. In addition, we obtain for the symmetric tensor $\mathbf{E}_{(\mu\nu)\alpha}$

$$\mathbf{E}_{(\mu\nu)\alpha} \approx \frac{\phi_0}{a},$$

and the typical correlation function

$$\int d\tau \langle x x(t) \rangle \approx \frac{k T a^2 \tau_v}{\phi_0}.$$

Thus we obtain for γ

$$\gamma \approx \sum \frac{\phi_0^2}{a^2} \frac{u^2}{L^2} \frac{k T a^2 \tau_v}{\phi_0} \approx \frac{k T \tau_v}{L^2} \sum \phi_0 u u \approx \frac{k T \tau_v \phi}{L^2},$$

where L corresponds to the typical deformation length with $L \ll \sigma$.

We are now prepared to evaluate the inelastic term in the fundamental Fokker-Planck equation for the distribution, $W \equiv W(\mathbf{r}^N, \mathbf{p}^N)$, which is a function of the granular degrees of freedom $\{\mathbf{r}^N, \mathbf{p}^N\}$. Let τ be an arbitrary time scale, and let the terms with asterisks be dimensionless $[O(1)]$. Given that O^* is a second order differential operator, the dissipative contribution will be [14]

$$\frac{1}{\tau} \frac{\partial W}{\partial t^*} \approx \frac{\gamma}{m^2 v_g^2} \frac{k T_g}{k T} O^* W.$$

Replacing the value obtained for γ, we have

$$\frac{1}{\tau} \frac{\partial W}{\partial t^*} \approx \frac{k T \tau_v \phi}{L^2} \frac{1}{m^2 v_g^2} \frac{k T_g}{k T} O^* W \approx \frac{\phi \tau_v}{m L^2} O^* W.$$

Letting $\tau_i \approx L / v_g$, where τ_i is the time a granular particle takes to move a distance corresponding to its deformation length ($\sim L$), we obtain

$$\frac{1}{\tau} \frac{\partial W}{\partial t^*} \approx \frac{\tau_v}{\tau_i^2} O^* W.$$

If we set the time scale $\tau = \tau_i$, we obtain

$$\frac{\partial W}{\partial t^*} \approx \left(\frac{\tau_v}{\tau_i} \right) O^* W. \tag{30}$$

So we observe that the order of magnitude of the dissipative term on the time scale of τ_i with respect to the streaming term is given roughly by the ratio between the granular velocity and the velocity of sound in the material that constitute the grains. The relative order of magnitude of the dissipative term in the Fokker-Planck equation for W relative to the streaming term is given approximatively by

$$\frac{\tau_v \tau_c}{\tau_i^2} \approx \frac{\sigma}{L} \frac{v_g}{v_s}.$$

For typical $v_{sound} \sim (10^3 - 10^4 v_g)$ and $L/\sigma \sim 10^{-2}$, one sees that the ratio is small as expected, but not extremely small.

In a recent paper [11], the authors propose a model which implies that in order to break the asperities of the surface of two grains (plastic deformation) in contact, the tangential stress (which is a function of shear deformation) has to exceed a material dependent threshold. In other words, the collision has to be energetic enough to conquer the elastic potential barrier. We believe then, that the present viscoelastic frictional model will describe satisfactorily slow collisions for rough and also smooth granular systems (in which case the energy dissipated by plastic deformations of asperities will be far smaller than that for typical systems found in nature).

VII. CONCLUSIONS

In the present study, we obtain the coefficient of instantaneous energy dissipation for collisions between two grains by an appropriate separation of the potential energy into internal, granular, and coupling parts and with the help of a first principles theory for granular distribution functions.

The form for the dissipative coefficient $\gamma(r)$ that we obtain is identical to the ones obtained elsewhere [10,11] assuming phenomenological viscoelastic dissipative coefficients. The agreement suggests that the phenomenological model is a plausible one to describe the interaction between GPs that do not involve plastic deformation of the grains. We are extending the present model in order to include the case of rough granular systems.

ACKNOWLEDGMENTS

One of us (W.A.M.M.) would like to thank CAPES (Brazilian funding agency) for financial support and C.Z. Liu for carefully reading the manuscript.

[1] P. K. Haff, J. Fluid Mech. **134**, 401 (1983).
[2] F. G. Bridges, A. Hatzes, and D. N. C. Lin, Nature **309**, 333 (1984).
[3] Reports from lecture by Professor Josef Eibl, Struct. Eng. **62 A**, 169 (1984).
[4] Y. Truji, T. Tanaka, and T. Ishida, Powder Technol. **71**, 239 (1992).
[5] P. C. Johnson and R. Jackson, J. Fluid Mech. **176**, 67 (1987).
[6] J. T. Jenkins and S. B. Savage, J. Fluid Mech. **130** 187 (1983); S. McNamara and W. K. Young, Phys. Fluids **A 5**, 34 (1993).
[7] H. M. Jaeger and S. R. Nagel, Science **255**, 1523 (1992); H. M. Jaeger, S. R. Nagel, and R. P. Behringer, Phys. Today **49**, (4), 32 (1996).
[8] P. Evesque and J. Rajchenbach, Phys. Rev. Lett. **62**, 44 (1988); J. Rajchenbach, Europhys. Lett. **16**, 149 (1991).
[9] C. V. Raman, Phys. Rev. **12**, 442 (1918).
[10] G. Kuwabara and K. Kono, Jpn. J. Appl. Phys. **26**, 1230 (1987).
[11] N. V. Brilliantov, F. Spahn, J.-M. Hertzsch, and T. Pöschel, Phys. Rev. **E 53**, 5382 (1996).
[12] N. W. Ashcroft and N. D. Mermin, *Solid State Physics* (Saunders College, Philadelphia 1976), p. 66.
[13] N. W. Aschcroft and N. D. Mermin, *Solid State Physics* (Ref. [12]), p. 424.
[14] J. Schofield and I. Oppenheim, Physica A **196**, 209 (1993).
[15] N. G. van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam 1992), p. 24.
[16] P. Ullersma, Physica **32**, 27 (1966); G. W. Ford, M. Kac, and P. Mazur, J. Math. Phys. **6**, 504 (1965).
[17] L. D. Landau and E. M. Lifshitz, *Theory of Elasticity*, 2nd English ed. (Pergamon, New York, 1970), pp. 3 and 36; A. E. H. Love, *Mathematical Theory of Elasticity*, 4th ed. (Cambridge University Press, Cambridge), pp. 193–200.
[18] L. D. Landau and E. M. Lifshitz, *Theory of Elasticity*, 2nd English ed. (Pergamon, New York, 1970), p. 36.
[19] L. D. Landau and E. M. Lifshitz, *Theory of Elasticity* (Ref. [18]), pp. 32 and 34.